Closure of the stellar hydrodynamic equations for arbitrary distributions

Rafael Cubarsi

Dept. Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
08034 Barcelona, Catalonia, Spain
E-mail:rcubarsi@ma4.upc.edu

Abstract

The closure problem of the stellar hydrodynamic equations is studied in a general case by describing the family of phase space density functions for which the collisionless Boltzmann equation is strictly equivalent to a finite subset of moment equations. The method is based on the use of maximum entropy distributions, which are afterwards generalised to phase space density functions depending on any isolating integral of motion in terms of a polynomial function of degree n in the velocities. Then, there is an independent set of velocity moments, up to an order n, so that the higher-order moments can be expressed in terms of the independent moments; the collisionless Boltzmann equation is given by a set of differential equations expressed from symmetric tensors of rank up to $n+1$; the independent moment equations are those of an order of up to $n+1$; and the hydrodynamic equations of an order higher than $n+1$ are redundant.

KEY WORDS: hydrodynamics – methods: analytical – stars: kinematics – galaxies: kinematics and dynamics – galaxies: statistics.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 60, 62, 85.

1 Introduction

In the current paper, the closure problem is studied in a more general way than in Cubarsi (2010a), hereafter Paper I, by describing the whole family of phase space density functions for which the collisionless Boltzmann equation is strictly equivalent to a finite subset of hydrodynamic equations. In addition, it is proven that the redundancy of the higher-order moment equations and the recurrence of the velocity moments are of similar nature. The method is based on the use of maximum entropy distributions, which allow an easy algebraic treatment. The maximum entropy functions are taken as a basis of square-integrable functions, to expand any arbitrary, non-exponential function in terms of a $n-$degree polynomial, as a convergent power series. For such a general family of functions, i.e., the phase space density functions depending on any isolating integral of motion expressed as polynomial function in the velocities, the equivalence between the first set of moment equations, up to order $n+1$, and the collisionless Boltzmann equation is proven.

2 Arbitrary function of $\mathcal{P}_n$

The closure of the moment equations is also valid for any arbitrary density function $f(\mathcal{P}_n)$, even if it is not a maximum entropy function. By following the same steps as in the Appendix B of Cubarsi (2007), it can be shown that any phase density $f(\mathcal{P}_n)$, which is a square-integrable function with respect to the weight $e^{\mathcal{P}_n}$ in Γ_V, can be expressed as the following convergent power series

$$f(\mathcal{P}_n) = \sum_{k=1}^{\infty} \gamma_{k-1}\, e^{k\,\mathcal{P}_n}, \tag{1}$$

where the coefficients γ_{k-1} are constant, and f depends on time, space, and velocity through $\mathcal{P}_n$, as in Eq. 9 of Paper I. If we assume that the distribution function is continuously differentiable in the interior of Γ_V, this generalised Fourier series can be integrated or differentiated term by term without loosing its convergence property.

Instead of repeating the full derivation of the above property, the velocity distribution function will be written in similar terms as in Cubarsi (2007), where it was shown that any arbitrary quadratic density function $f(Q+\sigma)$ could be expressed as a convergent series of the Gaussian functions $e^{-\frac{1}{2}(Q+\sigma)\,k}$ with $k \geq 1$. Let us assume an infinite velocity domain Γ_V. Since $\mathcal{P}_n$ is upper bounded, there exist a value ζ, which may depend on time and position, such that $\mathcal{P}_n < \zeta$ for all velocity $\mathbf{V} \in \Gamma_V$. Thus, we can write $\mathcal{P}_n = -\frac{1}{2}(\mathcal{Q}_n + \sigma)$ with $\mathcal{Q}_n = -2(\mathcal{P}_n - \zeta)$ a positive definite form, and $\sigma = -2\zeta$. Hence, we are in the appropriate conditions to show that any function $f(\mathcal{Q}_n + \sigma)$ can be expressed as a convergent power series in terms of $e^{-\frac{1}{2}(\mathcal{Q}_n+\sigma)}$. The case of a finite velocity domain is quite similar but with the corresponding changes concerning the domain of the variables. Thus, if $\zeta_1 < \mathcal{P}_n < \zeta_2$, then the variable $\tau = \frac{1}{2}(\mathcal{Q}_n+\sigma)$, defined in Cubarsi (2007), belongs to the interval $I = (-\zeta_2, -\zeta_1)$, and the variable $\eta = e^{-\tau}$ belongs to the interval $J = (e^{-\zeta_2}, e^{-\zeta_1})$.

On the other hand, due to the boundary conditions, Eq. 7 of Paper I, when the velocity $\mathbf{V}$ approaches the boundary domain Γ_V, then $\mathcal{P}_n \to -\infty$ and $f(\mathcal{P}_n) \to 0$. According to Eq. 4

of Paper I, since $f(\mathcal{P}_n) > 0$, f must be an increasing function of $\mathcal{P}_n$, so that

$$\frac{df(\mathcal{P}_n)}{d\mathcal{P}_n} > 0, \tag{2}$$

in the interior of the domain $\Gamma_r \times \Gamma_V$, for any fixed time.

3 Moment equations

By taking derivatives in Eq. 1,

$$\frac{df(\mathcal{P}_n)}{d\mathcal{P}_n} = \sum_{k=1}^{\infty} \gamma_{k-1}\, k\, e^{k\,\mathcal{P}_n}, \tag{3}$$

and bearing in mind Eq. 5 of Paper I, we have

$$\frac{Df(\mathcal{P}_n)}{Dt} = \sum_{k=1}^{\infty} \gamma_{k-1}\, e^{k\,\mathcal{P}_n} \frac{D(k\,\mathcal{P}_n)}{Dt} = 0. \tag{4}$$

Then, for any $k \neq 0$, Eqs. 12 and 16 of Paper I are also fulfilled with $\widetilde{\mathcal{P}}_n = k\,\mathcal{P}_n$, $\tilde{\lambda}_i = k\,\lambda_i$, and $\widetilde{\mathbf{\Lambda}}_i = k\,\mathbf{\Lambda}_i$; since $\mathcal{P}_n$ is linearly dependent on the tensor elements λ_i, and $\frac{D\mathcal{P}_n}{Dt}$ and $\mathbf{\Lambda}_i$ are similarly related. Then, each term of the above series satisfies the collisionless Boltzmann equation, so that a linear relationship, similar to Eq. 23 of Paper I, holds. Therefore, for each term of the series Eq. 4, we get the integrals

$$\mathcal{M}^{(k)}_{\alpha\beta\gamma} = \int_{\Gamma_V} \phi_{\alpha\beta\gamma} \frac{D(k\,\mathcal{P}_n)}{Dt} e^{k\,\mathcal{P}_n}\, d^3\mathbf{V} = 0; \quad \alpha+\beta+\gamma = l, \tag{5}$$

and we sum up, according to the coefficients of the series, so that we obtain

$$\sum_{k=1}^{\infty} \gamma_{k-1}\; \mathcal{M}^{(k)}_{\alpha\beta\gamma} = \int_{\Gamma_V} \phi_{\alpha\beta\gamma} \sum_{k=1}^{\infty} \gamma_{k-1} \frac{D(k\,\mathcal{P}_n)}{Dt} e^{k\,\mathcal{P}_n}\, d^3\mathbf{V}. \tag{6}$$

Hence, according to Eq. 4, we are led to the general expression of the moment equations, which generalises Eq. 18 of Paper I

$$\mathcal{M}_{\alpha\beta\gamma} = \int_{\Gamma_V} \phi_{\alpha\beta\gamma} \frac{Df(\mathcal{P}_n)}{Dt}\, d^3\mathbf{V} = 0. \tag{7}$$

For any order $l = \alpha+\beta+\gamma$, the $l-$order moment equation is obtained as linear combination of the moment equations associated with each term of the series Eq. 4,

$$\mathcal{M}_{\alpha\beta\gamma} = \sum_{k=1}^{\infty} \gamma_{k-1}\; \mathcal{M}^{(k)}_{\alpha\beta\gamma}. \tag{8}$$

4 Equivalence

The relationship between the hydrodynamic equations and the collisionless Boltzmann equation is now established by substitution of Eq. 20 of Paper I in Eq. 6,

$$\mathcal{M}_{\alpha\beta\gamma} = \int_{\Gamma_V} \phi_{\alpha\beta\gamma} \sum_{k=1}^{\infty} \gamma_{k-1} \left(\sum_{\iota+\mu+\nu\leq n+1} \Lambda_{\iota\mu\nu}\, \phi_{\iota\mu\nu} \right) k\, e^{k\,\mathcal{P}_n}\, d^3\mathbf{V}. \tag{9}$$

By reordering terms,

$$\mathcal{M}_{\alpha\beta\gamma} = \sum_{\iota+\mu+\nu\leq n+1} \Lambda_{\iota\mu\nu} \left[\int_{\Gamma_V} \phi_{\alpha\beta\gamma}\, \phi_{\iota\mu\nu} \left(\sum_{k=1}^{\infty} \gamma_{k-1}\, k\, e^{k\,\mathcal{P}_n} \right) d^3\mathbf{V} \right], \tag{10}$$

and bearing in mind Eq. 3, we may also write

$$\mathcal{M}_{\alpha\beta\gamma} = \sum_{\iota+\mu+\nu\leq n+1} \left[\int_{\Gamma_V} \phi_{\alpha\beta\gamma}\, \phi_{\iota\mu\nu}\, \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}\, d^3\mathbf{V} \right] \Lambda_{\iota\mu\nu}. \tag{11}$$

Therefore, if only orders $\alpha+\beta+\gamma \leq n+1$ are considered, we are led to a similar relationship as Eq. 21 of Paper I, but with the inner product calculated with the weight function given by Eq. 2, which will be notated with a Gram matrix $\widetilde{\mathbf{G}}_2$. The resulting integrals are some generalised velocity moments, so that when $f(\mathcal{P}_n)$ is a maximum entropy function then $f(\mathcal{P}_n) = \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}$ and the generalised moments become ordinary velocity moments. Thus we write

$$\mathcal{M}_{\alpha\beta\gamma} = \sum_{\iota+\mu+\nu\leq n+1} \widetilde{G}(\alpha\beta\gamma, \iota\mu\nu)\, \Lambda_{\iota\mu\nu};\ \alpha+\beta+\gamma \leq n+1. \tag{12}$$

The relationship between a finite set of hydrodynamic equations and the collisionless Boltzmann equation, given by Eqs. 23, 25, and 26 of Paper I, is now expressed as $\mathbf{M}_{n+1} = \widetilde{\mathbf{G}}_2 \mathbf{\Lambda}_{n+1}$, so that

$$\mathbf{\Lambda}_{n+1} = \mathbf{0}_{n+1} \iff \mathbf{M}_{n+1} = \mathbf{0}_{n+1}. \tag{13}$$

Therefore, for any density function depending on an n-degree polynomial function $\mathcal{P}_n$, there is a finite set of independent moment equations, for the orders $i = 0, 1, \ldots, n+1$, which is equivalent to the collisionless Boltzmann equation. Furthermore, a recurrence law for moment equations similar to Eq. 27 of Paper I, but with the weight function given by Eq. 2, is satisfied.

5 Polynomial coefficients

Similar to Cubarsi (2010b), but for the general case of a non-maximum entropy function, it is possible to prove the linear relationship between the coefficients of the polynomial function $\mathcal{P}_n$ and a finite set of extended velocity moments.

As in Cubarsi (2010b, Appendix A.1), but now for the general case of an arbitrary density function $f(\mathcal{P}_n)$, we compute the coefficients $\boldsymbol{\lambda}_k$, $1 \leq k \leq n$, of Eq. 12 of Paper I, in terms of an extended set of moment constraints, by integrating

$$\int_{\Gamma_V} \nabla_{\mathbf{V}} \left[(\mathbf{V})^m f(\mathcal{P}_n) \right] d^3\mathbf{V} = (\mathbf{0})^{n+1} \tag{14}$$

as a result of applying in the integration process the boundary conditions given by Eq. 7 of Paper I. The resulting Gramian system matrix $\widetilde{\mathbf{G}}_2$ is now a matrix of inner products associated with the basis $\phi_{\alpha\beta\gamma} = V_1^\alpha V_2^\beta V_3^\gamma$, with regard to the weight given by Eq. 2.
We write the integrand of Eq. 14 in Greek indices, according to Eq. 9 of Paper I, and we assume $\alpha + \beta + \gamma = m$, $0 \leq m \leq n-1$.
By taking the V_1-derivative, we have

$$\begin{aligned}
&\frac{\partial (V_1^\alpha V_2^\beta V_3^\gamma\, f(\mathcal{P}_n))}{\partial V_1} = \\
&= \alpha\, V_1^{\alpha-1} V_2^\beta V_3^\gamma\, f(\mathcal{P}_n) + V_1^\alpha V_2^\beta V_3^\gamma\, \frac{\partial \mathcal{P}_n}{\partial V_1} \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n} = \\
&= \alpha\, V_1^{\alpha-1} V_2^\beta V_3^\gamma f(\mathcal{P}_n) + V_1^\alpha V_2^\beta V_3^\gamma \sum_{\iota+\mu+\nu \leq n} \lambda_{\iota\mu\nu}\, \iota\, V_1^{\iota-1} V_2^\mu V_3^\nu \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}.
\end{aligned}$$

The last summation can be carried out from $\iota \geq 1$ instead of $\iota \geq 0$. Thus, by noting $\iota - 1$ as ι, we then have

$$\begin{aligned}
\frac{\partial (V_1^\alpha V_2^\beta V_3^\gamma\, f(\mathcal{P}_n))}{\partial V_1} &= \alpha\, V_1^{\alpha-1} V_2^\beta V_3^\gamma f(\mathcal{P}_n) + \\
+ V_1^\alpha V_2^\beta V_3^\gamma \sum_{\iota+\mu+\nu \leq n-1} &\lambda_{(\iota+1)\mu\nu}\, (\iota+1)\, V_1^\iota V_2^\mu V_3^\nu \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}.
\end{aligned} \tag{15}$$

Similarly, the other derivatives are

$$\begin{aligned}
\frac{\partial (V_1^\alpha V_2^\beta V_3^\gamma\, f(\mathcal{P}_n))}{\partial V_2} &= \beta\, V_1^\alpha V_2^{\beta-1} V_3^\gamma f(\mathcal{P}_n) + \\
+ V_1^\alpha V_2^\beta V_3^\gamma \sum_{\iota+\mu+\nu \leq n-1} &\lambda_{\iota(\mu+1)\nu}\, (\mu+1)\, V_1^\iota V_2^\mu V_3^\nu \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}, \\
\frac{\partial (V_1^\alpha V_2^\beta V_3^\gamma\, f(\mathcal{P}_n))}{\partial V_3} &= \gamma\, V_1^\alpha V_2^\beta V_3^{\gamma-1} f(\mathcal{P}_n) + \\
+ V_1^\alpha V_2^\beta V_3^\gamma \sum_{\iota+\mu+\nu \leq n-1} &\lambda_{\iota\mu(\nu+1)}\, (\nu+1)\, V_1^\iota V_2^\mu V_3^\nu \frac{df(\mathcal{P}_n)}{d\mathcal{P}_n}.
\end{aligned} \tag{16}$$

Then, if the above expressions are substituted into Eq. 14, by using the notation of Eq. 12, we get

$$
\begin{aligned}
-\alpha\, m_{(\alpha-1)\beta\gamma} &= \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}(\alpha\beta\gamma,\iota\mu\nu)\;(\iota+1)\,\lambda_{(\iota+1)\mu\nu},\\
-\beta\, m_{\alpha(\beta-1)\gamma} &= \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}(\alpha\beta\gamma,\iota\mu\nu)\;(\mu+1)\,\lambda_{\iota(\mu+1)\nu},\\
-\gamma\, m_{\alpha\beta(\gamma-1)} &= \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}(\alpha\beta\gamma,\iota\mu\nu)\;(\nu+1)\,\lambda_{\iota\mu(\nu+1)}.
\end{aligned}
\tag{17}
$$

According to this notation, all the moments with a negative index must be considered null. The elements of tensors $\boldsymbol{\lambda}_k$, $1 \le k \le n$, involved in Eq. 9 of Paper I (λ_0 is the normalisation factor), are explicitly obtained in terms of the generalised moments up to an order $2(n-1)$, as well as of the ordinary velocity moments up to an order $n-2$, by inverting the above system of equations:

$$
\begin{aligned}
\lambda_{(\alpha+1)\beta\gamma} &= \frac{-1}{\alpha+1} \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}^{-1}(\alpha\beta\gamma,\iota\mu\nu)\;\iota\; m_{(\iota-1)\mu\nu},\\
\lambda_{\alpha(\beta+1)\gamma} &= \frac{-1}{\beta+1} \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}^{-1}(\alpha\beta\gamma,\iota\mu\nu)\;\mu\; m_{\iota(\mu-1)\nu},\\
\lambda_{\alpha\beta(\gamma+1)} &= \frac{-1}{\gamma+1} \sum_{\iota+\mu+\nu\le n-1} \widetilde{G}^{-1}(\alpha\beta\gamma,\iota\mu\nu)\;\nu\; m_{\iota\mu(\nu-1)}.
\end{aligned}
\tag{18}
$$

The foregoing expressions are valid for $\alpha+\beta+\gamma \le n-1$, and $\widetilde{G}^{-1}(\alpha\beta\gamma,\iota\mu\nu)$ is the corresponding element of the inverse of the matrix $\widetilde{\mathbf{G}}_2$.
Once the polynomial coefficients are calculated, it is possible to express the higher-order generalised moments in terms of them, making use of Eq. 17, for $\alpha+\beta+\gamma \ge n$, by using the corresponding inner products $\langle \phi_{\alpha\beta\gamma}, \phi_{\iota\mu\nu} \rangle$ with the new weight, instead of $\widetilde{G}(\alpha\beta\gamma,\iota\mu\nu)$. Also, as commented in Cubarsi (2010b), for a given density function, the quantities $\boldsymbol{\lambda}_k$, $k \le n$, are univocally related to the minimum set of moments $\mathbf{m}_k$, $k \le n$, so that all the higher-order moments could be computed from the former set. However, the proven linear relationship makes use of an extended set of moments up to an order $2(n-1)$. Only for $n = 2$, the minimum and the extended set of moments match up. For this case, the moment recurrence was explicitly used in Cubarsi (2007) to derivate the redundancy of the higher-order moment equations. However, for $n > 2$, the existence of a general analytical relationship involving only the minimum set of moments should be further investigated.

6 Conclusion

The description of how the Galaxy relaxes towards a steady state is still a matter of debate, but there are two processes that likely play an important role: phase mixing and violent relaxation. Lynden-Bell (1967), in a seminal work, gave a statistical description of how a rapid fluctuating gravitational field produces a relaxation mechanism under the collisionless Boltzmann equation, which involves phase mixing, by changing the coarse-grained phase-space density near the phase point of each star, and violent relaxation, analogous to collisions in a gas, by changing the energy per unit mass of a star. Lynden-Bell's approach leads, for a non-degenerate stellar system, to a Maxwell-Boltzmann macroscopic distribution. Improvements to the previous approach (e.g. Chavanis et al. 1996, Chavanis 1998) take into account, among other aspects, the self confinement of the Galaxy, which is related to the incomplete relaxation problem due to the hypothesis of ergodicity; the maximum-entropy production principle to obtain a closure of the relaxation equation of diffusion type (non collisionless) for the coarse-grained distribution function; and the estimation of the diffusion current, which generalises the Chandrasekhar (1943) and Lynden-Bell (1967) equations. These and similar approaches, from a statistical viewpoint, and following the Jeans' direct problem, lead to the most probable distribution function for an equilibrium configuration of the Galaxy, and provide information about the functional form of the distribution function, or about the conserved quantities along the stellar motion, by leading to a distribution function that may take the form $f(\mathcal{P}_n)$.

In this stage, once the system has achieved relaxation, and according to Jeans' inverse problem, the situation can be reversed. It can be approached not from the statistical dynamics viewpoint but from analytical dynamics, by assuming the regularity conditions about the definition of the local standard of rest, the continuity and differentiability of its velocity, and the existence of higher-order velocity moments. We then ask under which circumstances the collisionless Boltzmann equation admits a solution of the form $f(\mathcal{P}_n)$. This is indeed the appropriate context to study the motion of the centroid and the admissible form of the potential function. Therefore, dissipative forces are not considered in the collisionless Boltzmann equation, but are indirectly connected with the functional form of the distribution function. This situation can be also tackled by using the stellar hydrodynamic equations, which explicitly involves the velocity moments. Then, the closure problem necessarily arises of how the infinite hierarchy of moment equations is related to the finite character of the collisionless Boltzmann equation.

In this context, the equivalence of Boltzmann and moment equations had already been investigated for $n = 2$. In this case, Eq. 17 of Paper I corresponds to the Chandrasekhar's (1942) system of equations, which allows us to obtain, under a time-dependent model and different symmetry hypotheses, a quite general solution for the collisionless Boltzmann equation, with the possibility of describing a stellar system with arbitrary mean velocity and orientation of velocity ellipsoid (Sanz-Subirana & Català-Poch 1987, Sala 1990, Juan-Zornoza & Sanz-Subirana 1991, Juan-Zornoza 1995). In this case, de Orús (1952) proved that if the Chandrasekhar's equations are fulfilled, the continuity equation and the Jean's equation are also satisfied. In the same vein, working from velocity moments up to fourth-order, Juan-Zornoza (1995) showed that Chandrasekhar's (1942) equations could be derived from

the first four hydrodynamic equations. A more general result, for generalised Schwarzschild distributions, was derived in Cubarsi (2007): the first four hydrodynamic equations, along with a moment recurrence relationship acting as closure condition, make the infinite hierarchy of hydrodynamic equations equivalent to the collisionless Boltzmann equation. Now, the above results have been generalised to any velocity distribution function depending on a polynomial function in the velocity variables.
As a conclusion, we summarise the main results of the paper. The following statements are equivalent:

(a) The velocity distribution depends on an integral of motion which is a polynomial function of degree n.

(b) There is an independent set of velocity moments, up to an order n, so that the higher-order moments can be expressed in terms of the independent moments.

(c) The collisionless Boltzmann equation is given by a set of differential equations expressed from symmetric tensors of rank up to $n + 1$.

(d) The independent moment equations are those of an order of up to $n + 1$.

(e) The hydrodynamic equations of an order higher than $n + 1$ are redundant.

References

Chandrasekhar S. 1942, Principles of Stellar Dynamics. Dover Publications Inc., New York
Chandrasekhar S. 1943, NYASA, 45, 131 (included in later versions of the book Principles of Stellar Dynamics)
Chavanis P.-H., Sommeria J., Robert R. 1996, ApJ, 471, 385
Chavanis P.-H. 1998, MNRAS, 300, 981
Cubarsi R. 2007, MNRAS 207, 380
Cubarsi R. 2010a, Closure of the stellar hydrodynamic equations for maximum entropy distributions. Dept. Matemàtica Aplicada IV (Barcelona: UPC e-prints)
Cubarsi R. 2010b, A&A, 510, A103
de Orús J.J. 1952, PhD Thesis, Collectanea Mathematica, Vol.V(2). University of Barcelona, Barcelona
Juan-Zornoza J.M., Sanz-Subirana J. 1991 Ap&SS, 185, 95
Juan-Zornoza J.M. 1995, PhD Thesis (ISBN: 84-475-0767-X). University of Barcelona, Barcelona
Lynden-Bell D. 1967, MNRAS 136, 101
Sala F., 1990, A&A, 235, 85
Sanz-Subirana J., Català-Poch M.A. 1987, in 10th ERAM of the IAU, Vol.4, 267